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On composition operators acting between Hardy and weighted Bergman spaces

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Abstract

We present a unified approach to some known and some new criteria for the boundedness and compactness of composition operators mapping a weighted Bergman space A^p_α into another weighted Bergman space A^q_β , where $q \geq p$ and $\alpha, \beta > -1$, also obtaining some asymptotic formulas for the essential norm. The results are also valid in the limit cases when at least one of the spaces is a Hardy space (i.e., when α or $\beta = -1$) and complement the existing results by various authors.

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1. Introduction and main results

Let \mathbb{D} denote the open unit disc in the complex plane and \mathbb{T} its boundary, the unit circle. We will use the notation $H(\mathbb{D})$ for the algebra of all analytic functions on \mathbb{D} .

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An *analytic self-map* φ of \mathbb{D} is a function in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Every such map induces the *composition operator* C_φ acting on $H(\mathbb{D})$, defined by $C_\varphi(f) = f \circ \varphi$. It is well known that every composition operator is a bounded linear operator on any of the standard Hardy and Bergman spaces of \mathbb{D} . This had essentially appeared already in Littlewood's paper [14]; see also [9, Chapter 1]. The first papers in the modern spirit of operators acting on function spaces were [16,20]. The monographs [6,22] present further developments and give an excellent overview of the subject up to the early or mid 1990's.

A composition operator acting on other spaces of analytic functions, or between two different spaces, need not be bounded. Thus, the first and most natural question that arises is that of characterizing all possible bounded operators in terms of their symbols. Such conditions can be of either geometric or analytic nature (see [6,19,24]). One can find many instances of this research in the literature, not only for the composition operators but also for the closely related multiplication operators and for the weighted composition operators that generalize both the composition and the multiplication operators (see, for example, [4,5] or [7,8]).

The purpose of this paper is to present different characterizations of both the bounded and the compact composition operators acting between two function spaces, each of which can be a Hardy space or a weighted Bergman space, all possible combinations being admitted. This complements part of the recent results on the more general weighted composition operators between two weighted Bergman spaces [7,8]. Our approach is somewhat different from the one taken by Čučković and Zhao. We hardly use any operator theory and, in addition to the generalized Carleson measures and Berezin transform, we also emphasize the role of the generalized Nevanlinna counting function in the spirit of Smith's work [24–26]. Also, our method allows us to cover the operators from Hardy into Bergman spaces, thus extending their results to this case as well. We now review the motivation and describe our main results.

Recall that a linear operator is said to be *compact* if it takes bounded sets to relatively compact sets. In the case of composition operators acting between two Banach spaces of analytic functions, this has several equivalent formulations. The study of compactness of composition operators began on the Hardy space H^2 in the pioneering work by Shapiro and Taylor [23], achieving its high point in a paper by Shapiro [21] where, among other results, a formula for the essential norm of C_φ acting on H^2 was found. An important ingredient, both there and in our study, is the use of $N_{\varphi,\gamma}$, the generalized Nevanlinna counting function associated with φ and defined as follows:

$$N_{\varphi,\gamma}(w) = \sum_{z \in \varphi^{-1}\{w\}} \left(\log \frac{1}{|z|} \right)^\gamma, \quad w \in \mathbb{D} \setminus \{\varphi(0)\}, \quad \gamma > 0,$$

where $z \in \varphi^{-1}\{w\}$ is repeated according to the multiplicity of the zero of $\varphi - w$ at z . (In [21], the case $\gamma = 1$ was considered.)

MacCluer and Shapiro [15] showed that C_φ is compact on the Bergman space A^p if and only if φ has no finite angular derivative at any point on the unit circle (see also [6]). Later on, bounded and compact composition operators from the weighted Bergman space A_α^p into A_β^q , $p \leq q$, were characterized in terms of the generalized Nevanlinna counting function by Riedl (the case $\alpha = \beta = -1$, understanding in this limit case that $A_{-1}^p = H^p$, the Hardy

space) and Smith in [19,24] (for $\alpha, \beta > -1$); note that this does not quite cover all the cases. Their results can be stated as follows.

Theorem A. *Let $0 < p \leq q < \infty$ and $-1 \leq \alpha, \beta < \infty$, and let φ be an analytic self-map of \mathbb{D} . Then $C_\varphi : A_\alpha^p \rightarrow A_\beta^q$ is bounded if and only if*

$$N_{\varphi, 2+\beta}(z) = O\left(\log \frac{1}{|z|}\right)^{q(2+\alpha)/p}, \quad |z| \rightarrow 1. \quad (1.1)$$

If this is the case, then C_φ is compact if and only if

$$N_{\varphi, 2+\beta}(z) = o\left(\log \frac{1}{|z|}\right)^{q(2+\alpha)/p}, \quad |z| \rightarrow 1. \quad (1.2)$$

The following global integral characterizations in order for $C_\varphi : A_\alpha^p \rightarrow A_\beta^q$, $p \leq q$, to be bounded or compact, is known.

Theorem B. *Let $0 < p \leq q < \infty$ and $-1 < \alpha, \beta < \infty$, and let φ be an analytic self-map of \mathbb{D} . Then $C_\varphi : A_\alpha^p \rightarrow A_\beta^q$ is bounded if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q(2+\alpha)/p+q} |\varphi'(z)|^q (1 - |z|^2)^{q+\beta} dA(z) < \infty. \quad (1.3)$$

If this is the case, then C_φ is compact if and only if

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q(2+\alpha)/p+q} |\varphi'(z)|^q (1 - |z|^2)^{q+\beta} dA(z) = 0. \quad (1.4)$$

Here $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$, $a \in \mathbb{D}$, denotes the automorphism of \mathbb{D} which interchanges the origin and the point a . Such a map is its own inverse and satisfies the fundamental identity

$$|\sigma'_a(z)| = \frac{1 - |\sigma_a(z)|^2}{1 - |z|^2} = \frac{1 - |a|^2}{|1 - \bar{a}z|^2}. \quad (1.5)$$

One of the goals of this study is to show, without appealing to operator theory, that conditions (1.1) and (1.2) are equivalent to (1.3) and (1.4), respectively; see Proposition 7 below. The following characterizations, which are in part due to Smith, see Theorem A above, of bounded composition operators mapping from H^p or A_α^p into A_β^q are obtained as a consequence.

Theorem 1. *Let $0 < p \leq q < \infty$, $-1 \leq \alpha < \infty$ and $-1 < \beta < \infty$, and let φ be an analytic self-map of \mathbb{D} . Then the following statements are equivalent:*

- (1) $C_\varphi : A_\alpha^p \rightarrow A_\beta^q$ is bounded;
- (2) $N_{\varphi, 2+\beta}(z) = O\left(\log \frac{1}{|z|}\right)^{q(2+\alpha)/p}, \quad |z| \rightarrow 1;$

- (3) $\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q(2+\alpha)/p+s} |\varphi'(z)|^s (1 - |z|^2)^{s+\beta} dA(z) < \infty, \quad 0 \leq s < \infty;$
 (4) $\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q(2+\alpha)/p+s} (1 - |\varphi(z)|^2)^s (1 - |z|^2)^\beta dA(z) < \infty, \quad 0 \leq s < \infty;$
 (5) μ is a bounded $2 + q/p(2 + \alpha)$ -Carleson measure, where $d\mu(z) = N_{\varphi, 2+\beta}(z) dA(z)$.

Here, as is usual, by a *bounded t -Carleson measure* we mean a positive Borel measure μ on \mathbb{D} such that

$$\sup_I \frac{\mu(S(I))}{|I|^t} < \infty, \quad 0 < t < \infty, \quad (1.6)$$

where $|I|$ denotes the arc length of a subarc I of \mathbb{T} ,

$$S(I) = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I, \quad 1 - |I| \leq |z| \right\}$$

is the Carleson box based at I , and the supremum is taken over all subarcs I of \mathbb{T} such that $|I| \leq 1$. Thus a bounded 1-Carleson measure is just a standard Carleson measure, as introduced in [2,3] (see also [9]).

One observes in Theorem 1 the appearance of the free non-negative parameter s . This should not be too surprising. If the condition (3) is satisfied for $s = 0$, then the conditions (3) and (4) are satisfied for all $s \geq 0$ by the Schwarz–Pick lemma. The proof of the other direction relies on characterizations of t -Carleson measures, given in Lemma C below. Moreover, a change of variable and Lemma C show that (3) with $s = 2$ is equivalent to (5). It is also worth noting that Čučković and Zhao [8] have proved the equivalence of (1) and (3) of Theorem 1 in the case $s = 0$ in a different manner.

If the target space is $H^q = A_{-1}^q$, the case which is excluded in Theorem 1, then the following characterizations, which are in part due to Riedl, are obtained.

Theorem 2. *Let $0 < p \leq q < \infty$ and $-1 \leq \alpha < \infty$, and let φ be an analytic self-map of \mathbb{D} . Then the following statements are equivalent:*

- (1) $C_\varphi : A_\alpha^p \rightarrow H^q$ is bounded;
 (2) $N_{\varphi,1}(z) = O\left(\left(\log \frac{1}{|z|}\right)^{q(2+\alpha)/p}\right), \quad |z| \rightarrow 1;$
 (3) $\sup_{a \in \mathbb{D}} \frac{1}{2\pi} \int_0^{2\pi} |\sigma'_a(\varphi(e^{i\theta}))|^{q(2+\alpha)/p} d\theta < \infty;$
 (4) $\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q(2+\alpha)/p+2} |\varphi'(z)|^2 \log \frac{1}{|z|} dA(z) < \infty;$
 (5) μ is a bounded $2 + q(2 + \alpha)/p$ -Carleson measure, where $d\mu(z) = N_{\varphi,1}(z) dA(z)$.

Recall that a positive Borel measure μ on \mathbb{D} is said to be a *vanishing (compact) t -Carleson measure* if

$$\lim_{|I| \rightarrow 0} \frac{\mu(S(I))}{|I|^t} = 0, \quad 0 < t < \infty.$$

The compactness results corresponding to Theorems 1 and 2 are stated as Theorems 3 and 4 below, respectively. As in the case of Theorem 1, the following result is in part due to Smith.

Theorem 3. Let $0 < p \leq q < \infty$, $-1 \leq \alpha < \infty$ and $-1 < \beta < \infty$, and let φ be an analytic self-map of \mathbb{D} . Then the following statements are equivalent:

- (1) $C_\varphi : A_\alpha^p \rightarrow A_\beta^q$ is compact;
- (2) $N_{\varphi, 2+\beta}(z) = o\left(\log \frac{1}{|z|}\right)^{q(2+\alpha)/p}$, $|z| \rightarrow 1$;
- (3) $\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q(2+\alpha)/p+s} |\varphi'(z)|^s (1 - |z|^2)^{s+\beta} dA(z) = 0$, $0 \leq s < \infty$;
- (4) $\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q(2+\alpha)/p+s} (1 - |\varphi(z)|^2)^s (1 - |z|^2)^\beta dA(z) = 0$, $0 \leq s < \infty$;
- (5) μ is a vanishing $2 + q(2 + \alpha)/p$ -Carleson measure, where $d\mu(z) = N_{\varphi, 2+\beta}(z) dA(z)$.

Čučković and Zhao have proved the equivalence of (1) and (3) of Theorem 3 in the case $p > 1$ and $s = 0$ in a different manner, see [8].

As in the case of Theorem 2, the following result is in part due to Riedl.

Theorem 4. Let $0 < p \leq q < \infty$ and $-1 \leq \alpha < \infty$, and let φ be an analytic self-map of \mathbb{D} . Then the following statements are equivalent:

- (1) $C_\varphi : A_\alpha^p \rightarrow H^q$ is compact;
- (2) $N_{\varphi, 1}(z) = o\left(\left(\log \frac{1}{|z|}\right)^{q(2+\alpha)/p}\right)$, $|z| \rightarrow 1$;
- (3) $\lim_{|a| \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} |\sigma'_a(\varphi(e^{i\theta}))|^{q(2+\alpha)/p} d\theta = 0$;
- (4) $\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q(2+\alpha)/p+2} |\varphi'(z)|^2 \log \frac{1}{|z|} dA(z) = 0$;
- (5) μ is a vanishing $2 + (q/p)(2 + \alpha)$ -Carleson measure, where $d\mu(z) = N_{\varphi, 1}(z) dA(z)$.

One expects naturally that the essential norm $\|C_\varphi\|_e$ of a composition operator C_φ mapping A_α^p boundedly into A_β^q should be expressed in terms of formulas derived from the conditions in Theorems 1–4. Recall that the *essential norm* $\|C_\varphi\|_e$ of a bounded operator C_φ is its distance (in the operator norm) from compact operators, that is:

$$\|C_\varphi\|_e = \inf_K \|C_\varphi - K\|,$$

where the infimum is taken over all admissible compact operators. In a landmark paper [21] in this field, Shapiro showed that the essential norm of C_φ mapping on H^2 equals to

$$\limsup_{|z| \rightarrow 1} \frac{N_{\varphi, 1}(z)}{\log \frac{1}{|z|}}.$$

Recall that $H^2 = A_{-1}^2$. Shapiro's result was later generalized by Poggi-Corradini [18] to some other cases as well: when $\alpha \in \{-1, 0, 1\}$, the essential norm of C_φ as a mapping

on A_α^2 equals to

$$\limsup_{|z| \rightarrow 1} \frac{N_{\varphi, 2+\alpha}(z)}{\left(\log \frac{1}{|z|}\right)^{2+\alpha}}.$$

The following result gives several formulas for the essential norm of $C_\varphi : A_\alpha^p \rightarrow A_\beta^q$ up to a constant multiple when the target space is not $H^q = A_{-1}^q$. We mention in passing the well known fact that if $p > q$ then every operator from the unweighted Bergman space A^p into A^q is compact.

Theorem 5. *Let $1 < p \leq q < \infty$, $-1 \leq \alpha < \infty$ and $-1 < \beta < \infty$, and let φ be an analytic self-map of \mathbb{D} . If $C_\varphi : A_\alpha^p \rightarrow A_\beta^q$ is bounded, then the following quantities are comparable:*

$$A = \|C_\varphi\|_e^q;$$

$$B = \limsup_{|z| \rightarrow 1} \frac{N_{\varphi, 2+\beta}(z)}{\left(\log \frac{1}{|z|}\right)^{q(2+\alpha)/p}};$$

$$C = \limsup_{|a| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q(2+\alpha)/p+s} |\varphi'(z)|^s (1 - |z|^2)^{s+\beta} dA(z), \quad 0 \leq s < \infty;$$

$$D = \limsup_{|a| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q/p(2+\alpha)+s} (1 - |\varphi(z)|^2)^s (1 - |z|^2)^\beta dA(z), \quad 0 \leq s < \infty.$$

In the case $s = 0$, Čučković and Zhao [8] have proved the comparability of A and C of Theorem 5 in a different manner.

Our last result deals with the case excluded in Theorem 5. But first a word about the notation. Throughout the paper, we write $A \lesssim B$ if there is a positive constant C (independent on A and B) such that $A \leq CB$. The symbol \gtrsim is defined in an analogous way. Similarly, $A \simeq B$ means that $A \lesssim B$ and $A \gtrsim B$, that is, the quotient of the two quantities is bounded and stays bounded away from zero.

Theorem 6. *Let $1 < p \leq q < \infty$ and $-1 \leq \alpha < \infty$, and let φ be an analytic self-map of \mathbb{D} . If $C_\varphi : A_\alpha^p \rightarrow H^q$ is bounded, then $\|C_\varphi\|_e^q \simeq A \simeq B \simeq C$, where*

$$A = \limsup_{|z| \rightarrow 1} \frac{N_{\varphi, 1}(z)}{\left(\log \frac{1}{|z|}\right)^{q(2+\alpha)/p}};$$

$$B = \limsup_{|a| \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} |\sigma'_a(\varphi(e^{i\theta}))|^{q(2+\alpha)/p} d\theta;$$

$$C = \limsup_{|a| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q(2+\alpha)/p+2} |\varphi'(z)|^2 \log \frac{1}{|z|} dA(z).$$

More specifically, $B \leq \|C_\varphi\|_e^q$ and, if $\alpha = -1$, then $\|C_\varphi\|_e^q \leq A$.

Earlier on, Contreras and Hernández-Díaz [4,5] obtained characterizations of bounded, compact, and other weighted composition operators between two different Hardy spaces, and Gorkin and MacCluer [12] calculated an asymptotic formula for the essential norm of a composition operator acting from H^p to H^q when $p > q$.

In the next section we introduce the necessary background material and establish several equivalent characterizations involving the generalized Nevanlinna counting function of an analytic self-map φ of \mathbb{D} . These results give an approach completely independent of operator theory to the conditions which appear in Theorems A and B, leading in part to proofs of Theorems 1–6 which are presented in Section 3.

2. Background material

We now recall the definitions of some classical spaces of analytic functions. For $p > 0$, the Hardy space H^p is the space of functions $f \in H(\mathbb{D})$ that satisfy

$$\|f\|_{H^p}^p = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty,$$

and, for $p > 0$ and $\alpha > -1$, the weighted Bergman space A_α^p consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p \left(\log \frac{1}{|z|} \right)^\alpha dA(z) < \infty,$$

where dA stands for the normalized Lebesgue area measure on \mathbb{D} . Equivalently, $f \in A_\alpha^p$ if and only if

$$\int_{\mathbb{D}} |f(z)|^p (1 - |z|)^\alpha dA(z) < \infty.$$

There are many reasons why the space H^p should be considered as the limit case of A_α^p as $\alpha \rightarrow -1^+$. One of them is that $\lim_{\alpha \rightarrow -1^+} \|f\|_{A_\alpha^p} = \|f\|_{H^p}$ (see [30] for a detailed proof). Our initial definition of the weighted Bergman space also allows for a similar reasoning, in view of the following generalization of the Littlewood–Paley formula due to Stanton (see [11,27]):

$$\|f\|_{H^p}^p = |f(0)|^p + \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dA(z). \quad (2.1)$$

Such generalizations were first used by Stanton (see [11,27]). The above formula had essentially been proved earlier by Yamashita (see Theorem 1 of [29]) but in a slightly different form. There is also an analogous formula:

$$\|f\|_{A_\alpha^p}^p \simeq |f(0)|^p + \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \left(\log \frac{1}{|z|} \right)^{\alpha+2} dA(z). \quad (2.2)$$

(See [24, Lemma 2.3], for example.) Thus, it is also natural to define $A_{-1}^p = H^p$ in view of these reasons.

The following characterization of bounded t -Carleson measures appears to be useful for our purposes. For a proof, see [1, Lemma 2.1; 17, Proposition 2.1].

Lemma C. *Let μ be a positive Borel measure on \mathbb{D} , $0 < t < \infty$ and $0 < \tau < \infty$. Then μ is a bounded t -Carleson measure if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left(\frac{(1 - |a|^2)^\tau}{|1 - \bar{a}w|^{1+\tau}} \right)^t d\mu(w) < \infty,$$

and the supremum above is comparable to the supremum in the definition of t -Carleson measures (1.6). Moreover, μ is a vanishing t -Carleson measure if and only if

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} \left(\frac{(1 - |a|^2)^\tau}{|1 - \bar{a}w|^{1+\tau}} \right)^t d\mu(w) = 0.$$

Shapiro [21] was apparently the first to use effectively the change of variable formula due to Stanton [11, 27] in the study of composition operators. This formula plays a key role in our proofs, and we state it as follows (see [6, Theorem 2.3]).

Lemma D. *If g is a positive measurable function on \mathbb{D} and φ is an analytic self-map of \mathbb{D} , then*

$$\int_{\mathbb{D}} (g \circ \varphi)(z) |\varphi'(z)|^2 \left(\log \frac{1}{|z|} \right)^\gamma dA(z) = \int_{\mathbb{D}} g(w) N_{\varphi, \gamma}(w) dA(w), \quad 0 < \gamma < \infty.$$

We need two more lemmas. The first one, by Essén, Shea, Stanton and Shapiro, see [11, 21], says that even if the generalized Nevanlinna counting function $N_{\varphi, \gamma}$ is not a subharmonic function, it has the sub-mean value property if $\gamma \geq 1$. From now on we denote by $A(0, r)$ the Euclidean disc of radius r centered at the origin.

Lemma E. *If $1 \leq \gamma < \infty$, φ is an analytic self-map of \mathbb{D} , $\varphi(0) \neq 0$ and $0 < r < |\varphi(0)|$, then*

$$N_{\varphi, \gamma}(0) \leq \frac{1}{r^2} \int_{A(0, r)} N_{\varphi, \gamma}(z) dA(z).$$

Our next lemma contains two simple inequalities which can be proved by straightforward estimates from elementary Calculus.

Lemma F. *Let $0 < r \leq t \leq 1$. Then the following inequalities hold:*

$$\log \frac{1}{t} \leq \frac{1}{r} (1 - t^2) \leq \frac{2}{r} \log \frac{1}{t}.$$

The next result plays an important role in the proofs of Theorems 1–6. It gives a connection between the “little-oh” and “big-Oh” conditions on the one hand and some global integral conditions on the other hand. Therefore, in view of Lemma C, it gives equivalent conditions in terms of Carleson measures.

Proposition 7. *Let $1 \leq \gamma < \infty$ and $-1 < t < \infty$, and let φ be an analytic self-map of \mathbb{D} . Then*

$$N_{\varphi, \gamma}(w) = O\left(\left(\log \frac{1}{|w|}\right)^t\right), \quad |w| \rightarrow 1, \quad (2.3)$$

if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_a(z)|^{2+t} N_{\varphi, \gamma}(z) \, dA(z) < \infty, \quad (2.4)$$

and moreover,

$$N_{\varphi, \gamma}(w) = o\left(\left(\log \frac{1}{|w|}\right)^t\right), \quad |w| \rightarrow 1, \quad (2.5)$$

if and only if

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_a(z)|^{2+t} N_{\varphi, \gamma}(z) \, dA(z) = 0. \quad (2.6)$$

More precisely,

$$\limsup_{|z| \rightarrow 1} \frac{N_{\varphi, \gamma}(z)}{\left(\log \frac{1}{|z|}\right)^t} \simeq \limsup_{|z| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_z(w)|^{2+t} N_{\varphi, \gamma}(w) \, dA(w). \quad (2.7)$$

Proof. Suppose that (2.3) is satisfied. Then there exists a positive constant C , depending only on r_0 , such that

$$\int_{\mathbb{D} \setminus A(0, r)} |\sigma'_a(z)|^{2+t} N_{\varphi, \gamma}(z) \, dA(z) \leq C \int_{\mathbb{D} \setminus A(0, r)} |\sigma'_a(z)|^{2+t} \left(\log \frac{1}{|z|}\right)^t \, dA(z)$$

for all $r \in (r_0, 1)$. Fix such an r . Then Lemma F and Forelli–Rudin estimates [13, Theorem 1.7] yield

$$\int_{\mathbb{D} \setminus A(0, r)} |\sigma'_a(z)|^{2+t} N_{\varphi, \gamma}(z) \, dA(z) \lesssim (1 - |a|^2)^{2+t} \int_{\mathbb{D}} \frac{(1 - |z|^2)^t}{|1 - \bar{a}z|^{2(2+t)}} \, dA(z) \simeq 1$$

and (2.4) follows.

Conversely, suppose that (2.4) is satisfied. Let $D(w, r)$ denote the pseudohyperbolic disc $\{z \in \mathbb{D} : |\sigma_w(z)| < r\}$. Since $|\sigma_w(\varphi(0))| \rightarrow 1$ as $|w| \rightarrow 1$, Lemma E and the fact $1 - |w| \simeq |1 - \bar{w}z|$ for $w \in D(z, \frac{1}{2})$, implies

$$\begin{aligned} N_{\varphi, \gamma}(w) &\leq 4 \int_{A(0, \frac{1}{2})} N_{\sigma_w \circ \varphi, \gamma}(u) \, dA(u) \\ &= 4 \int_{D(w, \frac{1}{2})} N_{\varphi, \gamma}(z) |\sigma'_w(z)|^2 \, dA(z) \\ &\simeq (1 - |w|^2)^t \int_{D(w, \frac{1}{2})} |\sigma'_w(z)|^{2+t} N_{\varphi, \gamma}(z) \, dA(z) \end{aligned} \quad (2.8)$$

for $|w|$ sufficiently close to 1, and (2.3) follows by Lemma F. Thus the first part of the assertion is proved.

We now prove the last part of the assertion. The equivalence of the conditions (2.5) and (2.6) can be proved in a similar manner. By (2.8), we have

$$\limsup_{|z| \rightarrow 1} \frac{N_{\varphi, \gamma}(z)}{\left(\log \frac{1}{|z|}\right)^t} \lesssim \limsup_{|z| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_z(w)|^{2+t} N_{\varphi, \gamma}(w) dA(w).$$

To prove the reverse inequality, let us write $A = \limsup_{|z| \rightarrow 1} N_{\varphi, \gamma}(z) \left(\log \frac{1}{|z|}\right)^{-t}$. Then, given $\varepsilon > 0$, there exists $r_\varepsilon \in (0, 1)$ such that $N_{\varphi, \gamma}(z) \left(\log \frac{1}{|z|}\right)^{-t} \leq A + \varepsilon$ for all $|z| \geq r_\varepsilon$. Therefore

$$\int_{\mathbb{D}} |\sigma'_a(z)|^{2+t} N_{\varphi, \gamma}(z) dA(z) \lesssim \frac{(1 - |a|^2)^{2+t}}{(1 - r_\varepsilon)^{4+2t}} \int_{A(0, r_\varepsilon)} N_{\varphi, \gamma}(z) dA(z) + A + \varepsilon$$

and it follows that

$$\limsup_{|a| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_a(z)|^{2+t} N_{\varphi, \gamma}(z) dA(z) \lesssim \limsup_{|z| \rightarrow 1} \frac{N_{\varphi, \gamma}(z)}{\left(\log \frac{1}{|z|}\right)^t}. \quad \square$$

The following proposition explains the appearance of the free non-negative parameter s in Theorems 1, 3 and 5. The proof relies on the well-known fact that $f \in A_\alpha^p$ if and only if $f' \in A_{p+\alpha}^p$. Namely,

$$\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) \simeq \int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^{p+\alpha} dA(z) + |f(0)|^p \quad (2.9)$$

for $0 < p < \infty$, $-1 < \alpha < \infty$ and for an analytic function f in \mathbb{D} . This is a standard result. The inequality in one direction is a classical result due to Hardy and Littlewood (see Theorem 5.6 in [9]), while the reverse inequality can easily be proved by the methods used in [9, Chapter 5].

Proposition 8. *Let $0 < t, s < \infty$ and $-1 < \beta < \infty$, and let φ be an analytic self-map of \mathbb{D} . Then, for any $a \in \mathbb{D}$, the following quantities are comparable:*

$$\begin{aligned} A &= \int_{\mathbb{D}} |(\sigma'_a \circ \varphi)(z)|^t (1 - |z|^2)^\beta dA(z); \\ B &= \int_{\mathbb{D}} |(\sigma'_a \circ \varphi)(z)|^{t+s} (1 - |\varphi(z)|^2)^s (1 - |z|^2)^\beta dA(z) + |\sigma'_a(\varphi(0))|^t; \\ C &= \int_{\mathbb{D}} |(\sigma'_a \circ \varphi)(z)|^{t+s} |\varphi'(z)|^s (1 - |z|^2)^{\beta+s} dA(z) + |\sigma'_a(\varphi(0))|^t. \end{aligned}$$

Moreover, the factor $1 - |z|$ in the quantities A , B and C can be replaced by $-\log |z|$.

Proof. By the Schwarz–Pick lemma and the fundamental identity (1.5), we have

$$|\sigma'_a(\varphi(z))| \cdot |\varphi'(z)|(1 - |z|^2) \leq |\sigma'_a(\varphi(z))|(1 - |\varphi(z)|^2) = (1 - |\sigma_a(\varphi(z))|^2) \leq 1$$

and therefore $C \leq B \leq A + |\sigma'_a(\varphi(0))|^t$. Since $|(\sigma'_a \circ \varphi)(z)|^t$ is subharmonic in \mathbb{D} , it is easy to see that $|\sigma'_a(\varphi(0))|^t \lesssim A$, and we obtain $C \leq B \lesssim A$.

To prove $A \simeq C$, we apply formula (2.9) to the function $f(z) = ((\sigma'_a \circ \varphi)(z))^b$, where $b = t/s$. This gives

$$\begin{aligned} & \int_{\mathbb{D}} |(\sigma'_a \circ \varphi)(z)|^t (1 - |z|^2)^\beta dA(z) \\ &= \int_{\mathbb{D}} |((\sigma'_a \circ \varphi)(z))^b|^{t/b} (1 - |z|^2)^\beta dA(z) \\ &\simeq b^{t/b} 2^{t/b} |a|^{t/b} \int_{\mathbb{D}} \left(\frac{(1 - |a|^2)^{b/(1+b)}}{|1 - \bar{a}\varphi(z)|^{1+(b/(1+b))}} \right)^{t+t/b} \\ &\quad \times |\varphi'(z)|^{t/b} (1 - |z|^2)^{\beta+(t/b)} dA(z) + |\sigma'_a(\varphi(0))|^t, \end{aligned}$$

from which Lemmas D and C yield $A \simeq C$. The well-known fact

$$\|f\|_{A_x^p}^p \simeq \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z)$$

together with the reasoning above shows that $1 - |z|^2$ can be replaced by $-\log |z|$ in the assertion. \square

We will also make use of the following lemma. For the proof, see [10,13], or the original source [28].

Lemma G. Let $0 < p < \infty$, $-1 < \alpha < \infty$ and $f \in A_x^p$. Then

$$|f(z)|(1 - |z|^2)^{(2+\alpha)/p} \leq \left((1 + \alpha) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) \right)^{1/p}, \quad z \in \mathbb{D},$$

with equality only for the constant multiples of the function $f_a(z) = (-\sigma'_a(z))^{(2+\alpha)/p}$.

3. Proofs

Proofs of Theorems 1 and 3. By Theorem A and Propositions 7 and 8, conditions (1)–(4) are equivalent. An application of Lemmas C and D to the case when $s = 2$ in (3) shows that the measure μ such that $d\mu(z) = N_{\varphi, 2+\beta}(w) dA(w)$ is a bounded $2 + q(2 + \alpha)/p$ -Carleson measure, that is, condition (3) is equivalent to (5). Thus Theorem 1 is proved. A similar argument yields the proof of Theorem 3. \square

Proofs of Theorems 2 and 4. By Theorem A, Proposition 7, and Lemmas C and D, the conditions (1), (2), (4), and (5) are equivalent. There are many ways to conclude that the

condition (3) must be equivalent to the others. One way is to use functions $f_a = (-\sigma'_a)^{(2+\alpha)/p}$ to show that (1) implies (3) and then it remains to show that (3) implies one of the other conditions to complete the proof. However, we may apply the argument used in the proof of Proposition 8 to see that the conditions (3) and (4) are equivalent. Since

$$\|f\|_{H^2}^2 = 2 \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z) + |f(0)|^2$$

for an analytic function f on \mathbb{D} , an application of this fact to the function $f = (-\sigma'_a)^{q(2+\alpha)/2p}$ together with Lemmas D and C show that (3) and (4) are equivalent. We omit the details which are very much similar to those in the proof of Proposition 8. Theorem 2 is now proved. Theorem 4 can be proved in a similar manner. \square

Proof of Theorem 5. By Propositions 7 and 8, the quantities B , C and D are comparable. We will now prove $C \lesssim A \lesssim B$ to complete the proof.

To prove $C \lesssim A$, consider the functions $f_a = (-\sigma'_a)^{(2+\alpha)/p}$ for which $\|f\|_{A_\alpha^p} \simeq 1$ by Lemma G, and $f_a \rightarrow 0$ uniformly in compact subsets of \mathbb{D} as $|a| \rightarrow 1$. If $K : A_\alpha^p \rightarrow A_\beta^q$ is compact, then

$$\begin{aligned} \|C_\varphi - K\| &\geq \limsup_{|a| \rightarrow 1} \|C_\varphi(f_a) - Kf_a\|_{A_\beta^q} \\ &\geq \limsup_{|a| \rightarrow 1} \|C_\varphi(f_a)\|_{A_\beta^q} - \limsup_{|a| \rightarrow 1} \|Kf_a\|_{A_\beta^q} \\ &= \limsup_{|a| \rightarrow 1} \|C_\varphi(f_a)\|_{A_\beta^q}. \end{aligned}$$

Moreover, it also follows that

$$\|C_\varphi\|_e^q \geq \limsup_{|a| \rightarrow 1} \int_{\mathbb{D}} |\sigma'_a(\varphi(z))|^{q(2+\alpha)/p} \left(\log \frac{1}{|z|} \right)^\beta dA(z)$$

and, thus, $C \lesssim A$ is proved for $s = 0$.

It remains to show that $A \lesssim B$. To this end, let $C_\varphi : A_\alpha^p \rightarrow A_\beta^q$ be bounded and suppose

$$\limsup_{|z| \rightarrow 1} \frac{N_{\varphi, 2+\beta}(z)}{\left(\log \frac{1}{|z|} \right)^{q(2+\alpha)/p}} = \gamma > 0$$

(if $\gamma = 0$, there is nothing to prove by Theorem 3). Then there exists $r \in (0, 1)$ such that

$$\frac{N_{\varphi, 2+\beta}(z)}{\left(\log \frac{1}{|z|} \right)^{q(2+\alpha)/p}} \leq 2\gamma \quad (3.1)$$

for $|z| \geq r$. For an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on \mathbb{D} , let

$$T_n f(z) = \sum_{k=0}^n a_k z^k, \quad R_n f(z) = \sum_{k=n+1}^{\infty} a_k z^k$$

(T_n for Taylor and R_n for the remainder). Then $T_n : A_\alpha^p \rightarrow A_\beta^q$ is compact, and

$$\|C_\varphi\|_e = \|C_\varphi(T_n + R_n)\|_e \leq \|C_\varphi T_n\|_e + \|C_\varphi R_n\|_e = \|C_\varphi R_n\|_e \leq \|C_\varphi R_n\|,$$

from which we deduce that $\|C_\varphi\|_e \leq \liminf_{n \rightarrow \infty} \|C_\varphi R_n\|$. Hence, by (2.2) and Lemma D,

$$\begin{aligned} \|C_\varphi\|_e^q &\leq \liminf_{n \rightarrow \infty} \|C_\varphi R_n\|^q = \liminf_{n \rightarrow \infty} \sup_{\|f\|_{A_\alpha^p} \leq 1} \|C_\varphi R_n f\|_{A_\beta^q}^q \\ &\simeq \liminf_{n \rightarrow \infty} \sup_{\|f\|_{A_\alpha^p} \leq 1} \int_{\mathbb{D}} |R_n f(\varphi(z))|^{q-2} |(R_n f \circ \varphi)'(z)|^2 \left(\log \frac{1}{|z|}\right)^{2+\beta} dA(z) \\ &= \liminf_{n \rightarrow \infty} \sup_{\|f\|_{A_\alpha^p} \leq 1} \int_{\mathbb{D}} |R_n f(\varphi(z))|^{q-2} |R_{n-1} f'(\varphi(z))|^2 |\varphi'(z)|^2 \\ &\quad \times \left(\log \frac{1}{|z|}\right)^{2+\beta} dA(z) \\ &= \liminf_{n \rightarrow \infty} \sup_{\|f\|_{A_\alpha^p} \leq 1} \int_{\mathbb{D}} |R_n f(w)|^{q-2} |R_{n-1} f'(w)|^2 N_{\varphi, 2+\beta}(w) dA(w), \end{aligned}$$

from which inequality (3.1) and formula (2.2) yield

$$\begin{aligned} \|C_\varphi\|_e^q &\lesssim \gamma \liminf_{n \rightarrow \infty} \sup_{\|f\|_{A_\alpha^p} \leq 1} \int_{\mathbb{D}} |R_n f(w)|^{q-2} |R_{n-1} f'(w)|^2 \left(\log \frac{1}{|w|}\right)^{q(2+\alpha)/p} dA(w) \\ &\simeq \gamma \liminf_{n \rightarrow \infty} \sup_{\|f\|_{A_\alpha^p} \leq 1} \|R_n f\|_{A_{q(2+\alpha)/p}^q}^q. \end{aligned}$$

Applying Lemmas G and F we finally obtain

$$\|C_\varphi\|_e^q \lesssim \gamma \liminf_{n \rightarrow \infty} \sup_{\|f\|_{A_\alpha^p} \leq 1} \|R_n f\|_{A_\alpha^p}^q \leq \gamma \sup_{\|f\|_{A_\alpha^p} \leq 1} \|f\|_{A_\alpha^p}^q = \gamma,$$

which is what we wished to prove. \square

Proof of Theorem 6. By Proposition 7 and the proofs of Theorems 2 and 4, the quantities A , B and C are comparable. Since the inequality $B \leq \|C_\varphi\|_e^q$ can be proved in a similar manner to the corresponding part of Theorem 5, we settle to prove $\|C_\varphi\|_e^q \leq A$ in the case $\alpha = -1$. To this end, let

$$\limsup_{|z| \rightarrow 1} \frac{N_{\varphi, 2+\beta}(z)}{\left(\log \frac{1}{|z|}\right)^{q(2+\alpha)/p}} = A.$$

Then, given $\varepsilon \in (0, 1)$, there exists $r_\varepsilon \in (0, 1)$ such that

$$\frac{N_{\varphi, 2+\beta}(z)}{\left(\log \frac{1}{|z|}\right)^{q(2+\alpha)/p}} \leq A + \varepsilon \quad (3.2)$$

for $|z| \geq r_\varepsilon$. By the proof of Theorem 5, we have $\|C_\varphi\|_e \leq \liminf_{n \rightarrow \infty} \|C_\varphi R_n\|$, and since

$$\liminf_{n \rightarrow \infty} \sup_{\|f\|_{H^p} \leq 1} \int_{\Delta(0,r)} |R_n f(w)|^{q-2} |R_{n-1} f'(w)|^2 N_{\varphi,1}(w) dA(w) = 0$$

for any $r \in (0, 1)$, formulas (2.1) and (3.2) yield

$$\begin{aligned} \|C_\varphi\|_e^q &\leq \frac{q^2}{2} \liminf_{n \rightarrow \infty} \sup_{\|f\|_{H^p} \leq 1} \int_{\mathbb{D} \setminus \Delta(0,r_\varepsilon)} |R_n f(w)|^{q-2} |R_{n-1} f'(w)|^2 N_{\varphi,1}(w) dA(w) \\ &\leq \frac{q^2}{2} (A + \varepsilon) \liminf_{n \rightarrow \infty} \sup_{\|f\|_{H^p} \leq 1} \int_{\mathbb{D} \setminus \Delta(0,r_\varepsilon)} |R_n f(w)|^{q-2} |R_{n-1} f'(w)|^2 \\ &\quad \times \left(\log \frac{1}{|w|} \right)^{q/p} dA(w). \end{aligned}$$

Applying the well known inequality $|f(z)|(1 - |z|^2)^{1/p} \leq \|f\|_{H^p}$ [9, Chapter 8, Exercise 4] and Lemma F, we finally obtain

$$\|C_\varphi\|_e^q = \frac{A + \varepsilon}{r_\varepsilon^{q/p-1}} \liminf_{n \rightarrow \infty} \sup_{\|f\|_{H^p} \leq 1} \|R_n f\|_{H^p}^q \leq \frac{A + \varepsilon}{r_\varepsilon^{q/p-1}}.$$

Since $\varepsilon \rightarrow 0$ as $r_\varepsilon \rightarrow 1$, it follows that $\|C_\varphi\|_e^q \leq A$. \square

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